Math 279 Lecture 5 Notes

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1 Integration With Respect to Rough Functions

1.1 Lyons' approach and Chen's relation

Today, we try to solve ODEs of the form $\dot{y} = V(y)\dot{x}$, where V is a $\mathcal{C}^2$ function, and x is a Hölder continuous function, say, of exponent α. If $\alpha > 1/2$, we can study this ODE by first making sense of integrals of the form $\int_0^t V(y(\theta))\, dx(\theta)$. We develop a strategy to deal with such integrals when $1/3 < \alpha \leq 1/2$. Let's explain the idea first.

We wish to make sense of

$$y(t) - y(s) = \int_s^t V(y(\theta))\, dx(\theta).$$

To make sense of the right hand side, we may try the following approximation for small $t - s$:

$$\int_s^t V(y(\theta))\, dx(\theta) \approx V(y(s)) \underbrace{\int_s^t dx(\theta)}_{|t-s|^\alpha} + O(|t-s|^{2\alpha}).$$

If we have a very fine mesh for defining our integral, then $\sum_i |t_i - t_{i+1}|^{2\alpha}$ is small only when $2\alpha > 1$. This suggests a finer Taylor expansion of the form

$$\begin{aligned}\int_s^t V(y(\theta))\, dx(\theta) &= \int_s^t [V(y(s)) + DV(y(s))(y(\theta) - y(s))]\, dx(\theta) + O(|t-s|^{3\alpha}) \\ &= \int_s^t [V(y(s)) + DV(y(s))V(y(s))(x(\theta) - x(s))\, dx(\theta)] + O(|t-s|^{3\alpha}) \\ &= V(y(s)) \int_s^t dx(\theta) + DV(y(s))V(y(s)) \int_s^t (x(\theta) - x(s))\, dx(\theta) \\ &\quad + O(|t-s|^{3\alpha})\end{aligned}$$

To make this work, we still need to make sense of

$$\int_s^t (x(\theta) - x(s)) \otimes dx(\theta) = \left[\int_s^t (x^i(\theta) - x^i(s))\, dx^j(\theta) \right]_{i,j=1}^{\ell}.$$

Terry Lyons' idea in 1990 was to choose a candidate for $\mathbb{X}(s,t) = \int_s^t (x(\theta) - x(s)) \otimes dx(\theta)$, and given $(x(\cdot), \mathbb{X}(\cdot,\cdot))$, we can make sense of integrals of the form $\int_s^t V(y(\theta))\, dx(\theta)$. For example, given $(x, \mathbb{X})$, we can define

$$\mathscr{I}(\mathbf{x}) = \int_0^T F(\mathbf{x})\, d\mathbf{x}$$

for any C^2 function F, with $\mathscr{I}(\mathbf{x})$ continuous in $\mathbf{x}$.

Theorem 1.1 (Lyons-Victoire)**.** *Given $x \in \mathcal{C}^\alpha$, there exists a function $z \in \mathcal{C}^\alpha$ such that $z(0) = 0$ and*

$$|z(t) - z(s) - x(s) \otimes (x(t) - x(s))| \leq x_0 [x]_\alpha^2 |t-s|^{2\alpha}.$$

Here, $[x]_\alpha = \sup_{s \neq t \in [0,1]} \frac{|x(t)-x(s)|}{|t-s|^\alpha}$.

Here, we want to think of

$$z(t) = \int_0^t x(\theta) \otimes \, dx(\theta),$$

so that

$$z(t) - z(s) = \int_s^t x(\theta) \otimes \, dx(\theta).$$

We also want to think of

$$z(t) - z(s) - x(s) \otimes (x(t) - x(s)) = \mathbb{X}(s,t).$$

Let us write $x(s,t) = x(t) - x(s)$, so that we can write

$$z(s,t) := z(t) - z(s) = \mathbb{X}(s,t) + x(s) \otimes x(s,t).$$

From $s < u < t \implies z(s,u) + z(u,t) = z(s,t)$, we learn that $\mathbb{X}(s,t)$ must satisfy the following formula, known as **Chen's relation**:

$$\mathbb{X}(s,u) + \mathbb{X}(u,t) = \mathbb{X}(s,t) + [x(s) \otimes x(s,t) - x(s) \otimes x(s,u) - x(u) \otimes x(u,t)]$$

Using $x(s,t) = x(s,u) + x(s,t)$, we get

$$= \mathbb{X}(s,t) + x(s,u) \otimes x(u,t).$$

We can now define

$$[(x(\cdot), \mathbb{X}(\cdot,\cdot))]_\alpha := [x]_\alpha + \sup_{s \neq t \in [0,T]} \frac{|\mathbb{X}(s,t)|}{|t-s|^{2\alpha}}.$$

Remark 1.1 (Geometric Rough Path)**.** Roughly, $\dot{z}^{i,j} = x^i \dot{x}^j$. Then

$$\dot{z}^{ij} + \dot{z}^{ji} = x^i \dot{x}^j + x^j \dot{x}^i = \frac{d}{dt}(x^i x^j).$$

If the product rule applies, we expect

$$z^{ij}(s,t) + z^{ji}(s,t) = x^i(t)x^j(t) - x^i(s)x^j(s).$$

In general, this may not be true. For example, Itô calculus is not geometric, while Stratonovich calculus is geometric.

1.2 Convergence of the integral

Theorem 1.2 (Lyons)**.** *Let $(x, \mathbb{X})$ be as above (Chen's relation + $[(x,\mathbb{X})]_\alpha < \infty$), and let $F \in \mathcal{C}^2$. Then we can define*

$$\mathscr{I}(F) = \int_0^t F(\mathbf{x}) \cdot d\mathbf{x} = \lim_{|\pi| \to 0} \underbrace{\sum_i [F(x(t_i)) \cdot x(t_i, t_{i+1}) + DF(x(t_i))^* \mathbb{X}(t_, t_{i+1})]}_{\mathscr{R}(\pi)},$$

where $\pi = \{0 < t_1 < \cdots < t_n < t\}$ and $|\pi| = \max_i |t_{i+1} - t_i|$. Moreover,

$$\left| \int_s^t F(\mathbf{x}) \cdot d\mathbf{x} - (F(x(s)) \cdot x(s,t) + \underbrace{DF(x(s))^*}_{A(s)} \mathbb{X}(s,t)) \right| \leq c(\alpha) \|F\|_{C^2} [(x,\mathbb{X})]_\alpha^3 |t-s|^{3\alpha}.$$

Proof. Take a partition $\pi = \{s < t_0 < \cdots < t_{n-1} < t_n < t = t_{n+1}\}$. Pick some i, and compare $\mathscr{R}(\pi)$ with $\mathscr{R}(\pi - \{t_i\})$:

$$\begin{aligned}
\mathscr{R}(\pi) - \mathscr{R}(\pi - \{t_i\}) &= F(x(t_{i-1}))x(t_{i-1}, t_i) + F(x(t_i))x(t_i, t_{i+1}) \\
&\quad + A(t_{i-1})\mathbb{X}(t_{i-1}, t_i) + A(t_i)\mathbb{X}(t_i, t_{i+1}) \\
&\quad - F(x(t_{i-1}))x(t_{i-1}, t_{i+1}) + A(t_{i-1})\mathbb{X}(t_{i-1}, t_{i+1}) \\
&= y(t_{i-1}, t_i)x(t_i, t_{i+1}) + A(t_{i-1}, t_i)\mathbb{X}(t_i, t_{i+1}) \\
&\quad - A(t_{i-1})x(t_{i-1}, t_i) \otimes x(t_i, t_{i+1}) \\
&= [y(t_{i-1}, t_i)x(t_i, t_{i+1}) - A(t_{i-1})x(t_{i-1}, t_i) \otimes x(t_i, t_{i+1})] \\
&\quad + A(t_{i-1}, t_i)\mathbb{X}(t_i, t_{i+1}).
\end{aligned}$$

So we may estimate the error as

$$\begin{aligned}
|\mathscr{R}(\pi)| &= |[y(t_{i-1}, t_i)x(t_i, t_{i+1}) - A(t_{i-1})x(t_{i-1}, t_i) \otimes x(t_i, t_{i+1})] + A(t_{i-1}, t_i)\mathbb{X}(t_i, t_{i+1})| \\
&\leq \|F\|_{\mathcal{C}^2} |t_{i+1} - t_i|^{3\alpha} [x]_\alpha^3 + \|F\|_{\mathcal{C}^2} |t_{i+1} - t_{i-1}|^{3\alpha} \|F\|_{\mathcal{C}^2} [\mathbb{X}]_{2\alpha}.
\end{aligned}$$

Choose i so that $|t_{i+1} - t_i| \leq 2(t-s)/n$,

$$\mathscr{R}(\pi) - \mathscr{R}(\pi - \{t_i\}) \leq c_0 \frac{|t-s|^{3\alpha}}{n^{3\alpha}} 2^{3\alpha}.$$

Do this inductively to obtain

$$|\mathscr{R}(\pi) - \mathscr{R}(\varnothing)| \leq c_0 |t-s|^{3\alpha}.$$

From our proof, we can also deduce that $\mathscr{R}(\pi)$ converges as $|\pi| \to 0$. $\square$